Supplementary: Satisfaction of Assumption 2 - Linear Regimes of Tire Models

This supplementary describes how to ensure that Assumption 2 of the main text is satisfied by performing an offline verification on the computed zonotope reachable sets. Recall Assumption 2 is stated as:

Assumption 2. *The absolute values of the slip ratio and angle are bounded below their critical values (i.e., $|\lambda_f(t)|, |\lambda_r(t)| < \lambda^{cri}$ and $|\alpha_f(t)|, |\alpha_r(t)| < \alpha^{cri}$ hold for all time).*

Recall that in a Front-Wheel-Drive vehicle model, $F_{\mathrm{xr}}(t) = 0$ for all t as in Remark 3 in the main text. By plugging the front linear longitudinal tire model

$$F_{\mathrm{xf}}(t) = \frac{mgl_{\mathrm{f}}}{l}\mu\lambda_{\mathrm{f}}(t) \tag{1}$$

in the proposed longitudinal front tire force

$$F_{\mathrm{xf}}(t) = -mK_{v_{\mathrm{x}}}(v_{\mathrm{x}}(t) - v_{\mathrm{x}}^{\mathrm{des}}(t,p)) + m\dot{v}_{\mathrm{x}}^{\mathrm{des}}(t,p) - F_{\mathrm{xr}}(t) - mv_{\mathrm{y}}(t)r(t) + m\tau_{v_{\mathrm{x}}}(t,p), \tag{2}$$

one can derive the front slip ratio as

$$\lambda_{\mathrm{f}}(t) = \frac{l}{gl_{\mathrm{r}}\mu}\big(- K_{v_{\mathrm{x}}}v_{\mathrm{x}}(t) + K_{v_{\mathrm{x}}}v_{\mathrm{x}}^{\mathrm{des}}(t,p) + \dot{v}_{\mathrm{x}}^{\mathrm{des}}(t,p) - v_{\mathrm{y}}(t)r(t) + \tau_{v_{\mathrm{x}}}(t,p)\big). \tag{3}$$

Similarly by plugging the front linear lateral tire model

$$F_{\mathrm{yf}}(t) = c_{\alpha\mathrm{f}}\alpha_{\mathrm{f}}(t) \tag{4}$$

in the proposed lateral front tire force

$$F_{\mathrm{yf}}(t) = -\frac{I_{\mathrm{zz}}K_r}{l_{\mathrm{f}}}\left(r(t) - r^{\mathrm{des}}(t,p)\right) + \frac{I_{\mathrm{zz}}}{l_{\mathrm{f}}}\dot{r}^{\mathrm{des}}(t,p) - \frac{I_{\mathrm{zz}}K_h}{l_{\mathrm{f}}}\left(h(t) - h^{\mathrm{des}}(t,p)\right) + \frac{l_{\mathrm{r}}}{l_{\mathrm{f}}}F_{\mathrm{yr}}(t) + \frac{I_{\mathrm{zz}}}{l_{\mathrm{f}}}\tau_r(t,p), \tag{5}$$

one can derive the front slip angle as

$$\alpha_{\mathrm{f}}(t) = -\frac{I_{\mathrm{zz}}K_r}{l_{\mathrm{f}}c_{\alpha\mathrm{f}}}\left(r(t) - r^{\mathrm{des}}(t,p)\right) - \frac{I_{\mathrm{zz}}K_h}{l_{\mathrm{f}}c_{\alpha\mathrm{f}}}\left(h(t) - h^{\mathrm{des}}(t,p)\right) + \frac{I_{\mathrm{zz}}}{l_{\mathrm{f}}c_{\alpha\mathrm{f}}}\dot{r}^{\mathrm{des}}(t,p) + \frac{l_{\mathrm{r}}}{l_{\mathrm{f}}c_{\alpha\mathrm{f}}}F_{\mathrm{yr}}(t) + \frac{I_{\mathrm{zz}}}{l_{\mathrm{f}}c_{\alpha\mathrm{f}}}\tau_r(t,p). \tag{6}$$

If the slip ratio and slip angle computed in (3) and (6) satisfy Assumption 2, they achieve the proposed robust, partial feedback linearization controller as in (2) and (5).

By definition of zonotopes, any $\mathcal{R}_j = <c_{\mathcal{R}_j},\ G_{\mathcal{R}_j}>$, which is a zonotope reachable set computed by CORA under the hybrid vehicle model from a partition element introduced in Section VIII-B of our main text, can be bounded by a multi-dimensional box $\mathtt{int}(c_{\mathcal{R}_j} - |G_{\mathcal{R}_j}|\cdot\mathbf{1}, c_{\mathcal{R}_j} + |G_{\mathcal{R}_j}|\cdot\mathbf{1})$ where $\mathbf{1}$ is a column vector of ones. This multi-dimensional box gives interval ranges of all elements in the augmented vehicle state x^{aug} during the j-th time interval T_j, which allows us to conservatively estimate $\{|\alpha_{\mathrm{r}}(t)|\}_{t\in T_j}$, $\{F_{\mathrm{yr}}(t)\}_{t\in T_j}$ and $\{|\lambda_{\mathrm{f}}(t)|\}_{t\in T_j}$ via $\alpha_{\mathrm{r}}(t) = -\frac{v_{\mathrm{y}}(t)-l_{\mathrm{r}}r(t)}{v_{\mathrm{x}}(t)}$, $F_{\mathrm{yr}}(t) = c_{\alpha\mathrm{r}}\alpha_{\mathrm{r}}(t)$ and (3) respectively using Interval Arithmetic. The approximation of $\{F_{\mathrm{yr}}(t)\}_{t\in T_j}$ makes it possible to over-approximate $\{|\alpha_{\mathrm{f}}(t)|\}_{t\in T_j}$ via (6).

Note in (3) and (6) integral terms are embedded in $\tau_{v_{\mathrm{x}}}(t,p)$ and $\tau_r(t,p)$ as described in Section V-A of the main text. Because it is nontrivial to perform Interval Arithmetic over integrals, we extend x^{aug} to $x^{\mathrm{aug+}}$ by appending three more auxiliary states $\varepsilon_{v_{\rm x}}(t) := \int_{t_0}^{t} \|v_{\rm x}(s) - v_{\rm x}^{\rm des}(s,p)\|^2 ds$, $\varepsilon_r(t) := \int_{t_0}^{t} \|r(s) - r^{\rm des}(s,p)\|^2 ds$ and $\varepsilon_h(t) := \int_{t_0}^{t} \|h(s) - h^{\rm des}(s,p)\|^2 ds$. Notice

$$\begin{bmatrix} \dot{\varepsilon}_{v_{\rm x}}(t) \\ \dot{\varepsilon}_r(t) \\ \dot{\varepsilon}_h(t) \end{bmatrix} = \begin{bmatrix} \|v_{\rm x}(t) - v_{\rm x}^{\rm des}(t,p)\|^2 \\ \|r(t) - r^{\rm des}(t,p)\|^2 \\ \|h(t) - h^{\rm des}(t,p)\|^2 \end{bmatrix}, \tag{7}$$

then we can compute a higher-dimensional FRS of $x^{\rm aug+}$ during $[0, t_{\rm f}]$ through the same process as described in Section VI of the main text. This higher-dimensional FRS makes over-approximations of $\{\varepsilon_{v_{\rm x}}(t)\}_{t\in T_j}$, $\{\varepsilon_r(t)\}_{t\in T_j}$ and $\{\varepsilon_h(t)\}_{t\in T_j}$ available for computation in (3) and (6).

If the supremum of $\{|\lambda_{\rm f}(t)|\}_{t\in T_j}$ exceeds $\lambda^{\rm cri}$ or any supremum of $\{|\alpha_{\rm f}(t)|\}_{t\in T_j}$ and $\{|\alpha_{\rm r}(t)|\}_{t\in T_j}$ exceeds $\alpha^{\rm cri}$, then the corresponding partition section of $\mathcal{X}_0 \times \mathcal{P}$ may result in a system trajectory that violates Assumption 2. Therefore to ensure not-at-fault, we only run optimization over partition elements whose FRS outer-approximations satisfy Assumption 2. Finally we emphasize that such verification of Assumption 2 over each partition element that is described in Section VIII-B of the main text can be done offline.